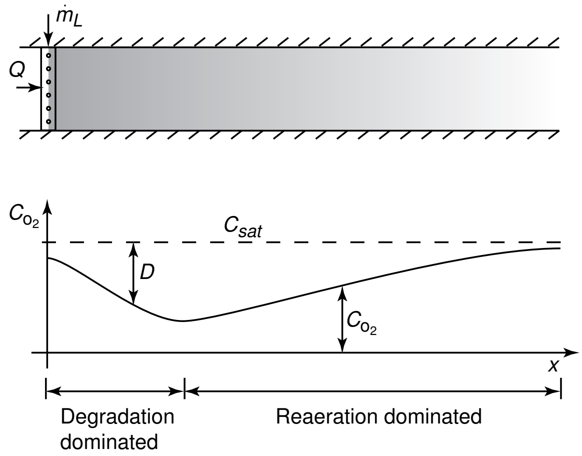
Exercise set 5

Problem 1 (Book chapter 5)



We study here the problem of biodegradation of waste in a stream. We have a flux of waste at the origin of the stream (see Figure above). As the waste is advected downstream, it degrades, thereby consuming oxygen. This oxygen deficit will be eventually restored by a counteracting aeration process. Find the concentration profile of O_2 in the stream under the following asssumptions

- The presence of biodegradable organic matter is described by a concentration of organic matter $[OM]$.
- The concentration of organic matter evolves according to a first order reaction, i.e. it follows the rate equation
$$\frac{d[OM]}{dt} = -k_d[OM] \Rightarrow [OM](t) = C^0_{[OM]}e^{-k_d t}. \tag{1}$$
Defining the oxygen deficit $D = [O_2^{\text{sat}}] - [O_2]$, we get from (1)
$$\frac{dD}{dt} = -\frac{d[OM]}{dt} = k_d C^0_{[OM]}e^{-k_d t}. \tag{2}$$
Note that what we denote $[OM]$ is actually a demand in oxygen (biochemical oxygen demand, i.e. the mass of oxygen required for degradation) and not directly the concentration of a particular substance.
- We further assume that the mass flux of oxygen $\dot{m}_{O_2}$ over an area A leading to reoxygenation is proportional to the oxygen deficit D
$$\dot{m}_{O_2} = -Ak_r D. \tag{3}$$

Combining this equation with (2) we get the equation for the evolution of the oxygen deficit taking into account two contributions (i.e. both the reoxygenation process and the degradation of waste)

$$\frac{dD}{dt} = \underbrace{k_d C^0_{[OM]} e^{-k_d t}}_{\text{non-homogeneous}} - \underbrace{K_r D}_{\text{homogeneous}} \quad . \tag{4}$$

We have introduced $K_r = k_r/h$ with h the depth of the flume since we are interested in quantities per unit volume. The solution of equation (4) gives us the concentration profile of O_2 downstream and is called the Streeter-Phelps equation (see also chapter 5 of the book by Socolofsky & Jirka).

Hint: Solve at first the homogeneous part of the equation. Then take a constant of integration varying in time and insert this solution in the non-homogeneous problem and solve it. Finally impose the boundary condition $D(t = 0) = D_0$.

Problem 2 (Book page 105)

Use the Streeter-Phelps equation to get the oxygen deficit along the stream, here we are interested in the dependence on the position along the stream and not in $D(t)$.

We consider the following parameters: $\dot{m} = 295\ g/s$ of BOD (biochemical oxygen demand, i.e. mass of oxygen required for degradation) into a stream $h = 3\ m$ deep, $W = 30\ m$ wide, and with a flow rate $Q = 27\ m^3/s$. Finally we measure $D_0 = 1.5\ mg/l$, $k_d = 0.2$ day^{-1} and $K_R = 0.4$ day^{-1}.